

# A CHARACTERIZATION OF BARYCENTRICALLY PREASSOCIATIVE FUNCTIONS

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**ABSTRACT.** We provide a characterization of the variadic functions which are barycentrically preassociative as compositions of length-preserving associative string functions with one-to-one unary maps. We also discuss some consequences of this characterization.

## 1. INTRODUCTION

Let  $X$  and  $Y$  be arbitrary nonempty sets. Throughout this paper we regard tuples  $\mathbf{x}$  in  $X^n$  as  $n$ -strings over  $X$ . We let  $X^* = \bigcup_{n \geq 0} X^n$  be the set of all strings over  $X$ , with the convention that  $X^0 = \{\varepsilon\}$  (i.e.,  $\varepsilon$  denotes the unique 0-string on  $X$ ). We denote the elements of  $X^*$  by bold roman letters  $\mathbf{x}, \mathbf{y}, \mathbf{z}$ . If we want to stress that such an element is a letter of  $X$ , we use non-bold italic letters  $x, y, z$ , etc. The *length* of a string  $\mathbf{x}$  is denoted by  $|\mathbf{x}|$ . For instance,  $|\varepsilon| = 0$ . We endow the set  $X^*$  with the concatenation operation, for which  $\varepsilon$  is the neutral element, i.e.,  $\varepsilon \mathbf{x} = \mathbf{x} \varepsilon = \mathbf{x}$ . For instance, if  $\mathbf{x} \in X^m$  and  $y \in X$ , then  $\mathbf{x}y \in X^{m+1}$ . Moreover, for every string  $\mathbf{x}$  and every integer  $n \geq 0$ , the power  $\mathbf{x}^n$  stands for the string obtained by concatenating  $n$  copies of  $\mathbf{x}$ . In particular we have  $\mathbf{x}^0 = \varepsilon$ .

As usual, a map  $F: X^n \rightarrow Y$  is said to be an *n-ary function* (an *n-ary operation on X* if  $Y = X$ ). Also, a map  $F: X^* \rightarrow Y$  is said to be a *variadic function* (a *string function on X* if  $Y = X^*$ ; see [5]). For every variadic function  $F: X^* \rightarrow Y$  and every integer  $n \geq 0$ , we denote by  $F_n$  the *n-ary part*  $F|_{X^n}$  of  $F$ .

Recall that a variadic function  $F: X^* \rightarrow Y$  is said to be *preassociative* [6, 7] if, for any  $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*$ , we have

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z}).$$

Also, a variadic function  $F: X^* \rightarrow Y$  is said to be *barycentrically preassociative* (or *B-preassociative* for short) [8] if, for any  $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*$ , we have

$$|\mathbf{y}| = |\mathbf{y}'| \quad \text{and} \quad F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z}).$$

Contrary to preassociativity, B-preassociativity recalls the associativity-like property of the barycenter (just regard  $F(\mathbf{x})$  as the barycenter of a set  $\mathbf{x}$  of identical homogeneous balls in  $X = \mathbb{R}^n$ ). In descriptive statistics and aggregation function theory, this condition says that the aggregated value of a series of numerical values remains unchanged when modifying a bundle of these values without changing their partial aggregation.

B-preassociativity has been recently utilized by the authors in the following characterization of the *quasi-arithmetic pre-mean functions*, thus generalizing the well-known Kolmogoroff-Nagumo's characterization of the quasi-arithmetic mean functions.

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**Theorem 1** ([8]). *Let  $\mathbb{I}$  be a nontrivial real interval, possibly unbounded. A function  $F: \mathbb{I}^* \rightarrow \mathbb{R}$  is B-preassociative and, for every  $n \geq 1$ , the function  $F_n$  is symmetric, continuous, and strictly increasing in each argument if and only if there are continuous and strictly increasing functions  $f: \mathbb{I} \rightarrow \mathbb{R}$  and  $f_n: \mathbb{R} \rightarrow \mathbb{R}$  ( $n \geq 1$ ) such that*

$$F_n(\mathbf{x}) = f_n\left(\frac{1}{n} \sum_{i=1}^n f(x_i)\right), \quad n \geq 1.$$

*Remark 1.* If we add the condition that every  $F_n$  is idempotent (i.e.,  $F_n(x^n) = x$  for every  $x \in X$ ) in Theorem 1, then we necessarily have  $f_n = f^{-1}$  for every  $n \geq 1$ , thus reducing this result to Kolmogoroff-Nagumo's characterization of the quasi-arithmetic mean functions [4, 9]. However, there are also many non-idempotent quasi-arithmetic pre-mean functions. Taking for instance  $f_n(x) = nx$  and  $f(x) = x$  over the reals  $\mathbb{I} = \mathbb{R}$ , we obtain the sum function. Taking  $f_n(x) = \exp(nx)$  and  $f(x) = \ln(x)$  over  $\mathbb{I} = ]0, \infty[$ , we obtain the product function.

In this paper we show that B-preassociative functions can be factorized as compositions of length-preserving associative string functions with one-to-one unary maps. We also show how this factorization result generalizes a characterization of a noteworthy subclass of B-preassociative functions given by the authors in [8]. Finally, we mention some interesting consequences of this new characterization.

The terminology used throughout this paper is the following. The domain, range, and kernel of any function  $f$  are denoted by  $\text{dom}(f)$ ,  $\text{ran}(f)$ , and  $\text{ker}(f)$ , respectively. The identity function on any nonempty set is denoted by  $\text{id}$ . For every  $n \geq 1$ , the diagonal section  $\delta_F: X \rightarrow Y$  of a function  $F: X^n \rightarrow Y$  is defined as  $\delta_F(x) = F(x^n)$ .

*Remark 2.* Although B-preassociativity was recently defined by the authors [8], the basic idea behind this definition goes back to 1931 when de Finetti [1] introduced an associativity-like property for mean functions. Indeed, according to de Finetti, for a real function  $F: \bigcup_{n \geq 1} \mathbb{R}^n \rightarrow \mathbb{R}$  to be considered as a mean, it is natural that it be “associative” in the following sense: for any  $u \in X$  and any  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$  such that  $|\mathbf{xz}| \geq 1$  and  $|\mathbf{y}| \geq 1$ , we have  $F(\mathbf{xyz}) = F(\mathbf{x}u^{|\mathbf{y}|}\mathbf{z})$  whenever  $F(\mathbf{y}) = F(u^{|\mathbf{y}|})$ .

## 2. MAIN RESULTS

As mentioned in the introduction, in this section we mainly show that B-preassociative functions can be factorized as compositions of length-preserving associative string functions with one-to-one unary maps. This result is stated in Theorem 8.

Recall that a string function  $F: X^* \rightarrow X^*$  is said to be *associative* [5] if it satisfies the equation  $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$  for any  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ .

**Definition 2.** We say that a string function  $F: X^* \rightarrow X^*$  is *length-preserving* if  $|F(\mathbf{x})| = |\mathbf{x}|$  for every  $\mathbf{x} \in X^*$ , or equivalently, if  $\text{ran}(F_n) \subseteq X^n$  for every  $n \geq 0$ .

Clearly, the identity function on  $X^*$  is associative and length-preserving. The following example gives nontrivial instances of associative and length-preserving string functions. Further examples of associative string functions can be found in [5].

**Example 3.** Let  $(h_n)_{n \geq 1}$  be a sequence of unary operations on  $X$ . One can easily see that the length-preserving function  $F: X^* \rightarrow X^*$  defined by  $F_0(\varepsilon) = \varepsilon$  and

$$F_n(x_1 \cdots x_n) = h_1(x_1) \cdots h_n(x_n), \quad n \geq 1,$$

is associative if and only if  $h_n \circ h_m = h_n$  for all  $n, m \geq 1$  such that  $m \leq n$ . Using an elementary induction, one can also show that the latter condition is equivalent to  $h_n \circ h_n =$

$h_n$  and  $h_{n+1} \circ h_n = h_{n+1}$  for every  $n \geq 1$ . To give an example, take any constant sequence  $h_n = h$  such that  $h \circ h = h$  (for instance, the positive part function  $h(x) = x^+$  over  $X = \mathbb{R}$ ). As a second example, consider the sequence  $h_n$  of unary operations on  $X = \{1, 2, 3, \dots\}$  defined by  $h_n(k) = 1$  if  $k \leq n + 1$ , and  $h_n(k) = k$ , otherwise.

**Proposition 4.** *Let  $F: X^* \rightarrow X^*$  be a length-preserving function. Then  $F$  is associative if and only if it is B-preassociative and satisfies  $F_n = F_n \circ F_n$  for every  $n \geq 0$ .*

*Proof.* To see that the necessity holds, we recall from [5] that any associative string function is preassociative and hence B-preassociative. The second part of the statement is immediate. For the sufficiency, we merely observe that we have  $F(F(y)) = F(y)$  for every  $y \in X^*$  and therefore, by B-preassociativity, we also have  $F(xyzy) = F(xyz)$  for every  $xyz \in X^*$ , that is,  $F$  is associative.  $\square$

The following proposition, established in [8], shows how we can construct new B-preassociative functions from given B-preassociative functions.

**Proposition 5** ([8]). *Let  $F: X^* \rightarrow Y$  be a B-preassociative function and let  $(g_n)_{n \geq 1}$  be a sequence of functions from  $Y$  to a nonempty set  $Y'$ . If  $g_n|_{\text{ran}(F_n)}$  is one-to-one for every  $n \geq 1$ , then any function  $H: X^* \rightarrow Y'$  such that  $H_n = g_n \circ F_n$  for every  $n \geq 1$  is B-preassociative.*

Recall that a function  $g$  is a *quasi-inverse* [10, Sect. 2.1] of a function  $f$  if

$$f \circ g|_{\text{ran}(f)} = \text{id}|_{\text{ran}(f)} \quad \text{and} \quad \text{ran}(g|_{\text{ran}(f)}) = \text{ran}(g).$$

We denote the set of quasi-inverses of a function  $f$  by  $Q(f)$ . Under the assumption of the Axiom of Choice (AC), the set  $Q(f)$  is nonempty for any function  $f$ . In fact, the Axiom of Choice is just another form of the statement “every function has a quasi-inverse”. Note also that the relation of being quasi-inverse is symmetric: if  $g \in Q(f)$  then  $f \in Q(g)$ ; moreover, we have  $\text{ran}(g) \subseteq \text{dom}(f)$  and  $\text{ran}(f) \subseteq \text{dom}(g)$  and the functions  $f|_{\text{ran}(g)}$  and  $g|_{\text{ran}(f)}$  are one-to-one.

**Lemma 6.** *Assume AC and let  $F: X^n \rightarrow Y$  be a function. For any  $g \in Q(F)$ , define the function  $H: X^n \rightarrow X^n$  by  $H = g \circ F$ . Then we have  $F = F \circ H$  and  $H = H \circ H$ . Moreover, the map  $F|_{\text{ran}(H)}$  is one-to-one.*

*Proof.* By definition of  $H$  we have  $F \circ H = F \circ g \circ F = F$  and  $H \circ H = g \circ F \circ g \circ F = g \circ F = H$ . Also, the map  $F|_{\text{ran}(H)} = F|_{\text{ran}(g)}$  is one-to-one.  $\square$

**Lemma 7.** *Assume AC and let  $F: X^* \rightarrow Y$  be a function. The following assertions are equivalent.*

- (i)  $F$  is B-preassociative.
- (ii) For every sequence  $(g_n \in Q(F_n))_{n \geq 1}$ , the function  $H: X^* \rightarrow X^*$  defined by  $H_0(\varepsilon) = \varepsilon$  and  $H_n = g_n \circ F_n$  for every  $n \geq 1$  is associative and length-preserving.
- (iii) There exists a sequence  $(g_n \in Q(F_n))_{n \geq 1}$  such that the function  $H: X^* \rightarrow X^*$  defined by  $H_0(\varepsilon) = \varepsilon$  and  $H_n = g_n \circ F_n$  for every  $n \geq 1$  is associative and length-preserving.

*Proof.* (i)  $\Rightarrow$  (ii). Let  $H: X^* \rightarrow X^*$  be defined as indicated in the statement. We know by Lemma 6 that  $H \circ H = H$  and  $H$  is length-preserving. Since  $g_n|_{\text{ran}(F_n)}$  is one-to-one, we have that  $H$  is B-preassociative by Proposition 5. It follows from Proposition 4 that  $H$  is associative.

(ii)  $\Rightarrow$  (iii). Trivial.

(iii)  $\Rightarrow$  (i). By Proposition 4,  $H$  is B-preassociative. For every  $n \geq 1$ , since  $g_n|_{\text{ran}(F_n)}$  is a one-to-one map from  $\text{ran}(F_n)$  onto  $\text{ran}(g_n) = \text{ran}(H_n)$ , we have  $F_n = (g_n|_{\text{ran}(F_n)})^{-1} \circ H_n$ . By Proposition 5 it follows that  $F$  is B-preassociative.  $\square$

We are now ready to present our main result, which gives a characterization of any B-preassociative function as a composition of a length-preserving associative string function with one-to-one unary maps.

**Theorem 8.** *Assume AC and let  $F: X^* \rightarrow Y$  be a function. The following assertions are equivalent.*

- (i)  $F$  is B-preassociative.
- (ii) *There exist an associative and length-preserving function  $H: X^* \rightarrow X^*$  and a sequence  $(f_n)_{n \geq 1}$  of one-to-one functions  $f_n: \text{ran}(H_n) \rightarrow Y$  such that  $F_n = f_n \circ H_n$  for every  $n \geq 1$ .*

*If condition (ii) holds, then for every  $n \geq 1$  we have  $f_n = F|_{\text{ran}(H_n)} = F_n|_{\text{ran}(H_n)}$ ,  $f_n^{-1} \in Q(F_n)$ , and we may choose  $H_n = g_n \circ F_n$  for any  $g_n \in Q(F_n)$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Let  $H: X^* \rightarrow X^*$  be defined by  $H_0(\varepsilon) = \varepsilon$  and  $H_n = g_n \circ F_n$  for every  $n \geq 1$ , where  $g_n \in Q(F_n)$ . By Lemma 6 we have  $F_n = f_n \circ H_n$  for every  $n \geq 1$ , where  $f_n = F_n|_{\text{ran}(H_n)}$  is one-to-one. By Lemma 7,  $H$  is associative and length-preserving.

(ii)  $\Rightarrow$  (i).  $H$  is B-preassociative by Proposition 4. By Proposition 5 it follows that also  $F$  is B-preassociative.

If condition (ii) holds, then for every  $n \geq 1$  we have  $F_n \circ H_n = f_n \circ H_n \circ H_n = f_n \circ H_n$  and hence  $F_n|_{\text{ran}(H_n)} = f_n$ . Moreover, since  $f_n$  is one-to-one, we have  $H_n = f_n^{-1} \circ F_n$  and hence  $F_n \circ f_n^{-1} \circ F_n = F_n \circ H_n = f_n \circ H_n \circ H_n = f_n \circ H_n = F_n$ , which shows that  $f_n^{-1} \in Q(F_n)$ .  $\square$

- Remark 3.* (a) It is clear that the trivial factorization  $F_n = F_n \circ H_n$ , where  $H_n = \text{id}$ , holds for any function  $F: X^* \rightarrow Y$ . This observation could make us wrongly think that Theorem 8 is of no use. However, in our factorization  $F_n = f_n \circ H_n$  the outer function  $f_n$  has the important feature that it is one-to-one.
- (b) Similarly to Theorem 8, one can show [5] that any preassociative function  $F: X^* \rightarrow Y$  can be factorized as a composition  $F = f \circ H$ , where  $H: X^* \rightarrow X^*$  is associative and  $f: \text{ran}(H) \rightarrow Y$  is one-to-one.

In the rest of this section we show how Theorem 8 can be particularized to some nested subclasses of B-preassociative functions, including the subclass of B-preassociative functions  $F: X^* \rightarrow Y$  for which the equality  $\text{ran}(F_n) = \text{ran}(\delta_{F_n})$  holds for every  $n \geq 1$  (see [8]).

For any integers  $m, n \geq 1$ , define  $X_m^0 = X^0$  and

$$X_m^n = \{yz^{n-\min\{n,m\}+1} : yz \in X^{\min\{n,m\}}\}.$$

For instance  $X_1^3 = \{z^3 : z \in X\}$ ,  $X_2^3 = \{yz^2 : yz \in X^2\}$ , and  $X_m^3 = X^3$  for every  $m \geq 3$ .

Thus, we have  $X_m^n = X^n$  if  $m \geq n$  and  $X_m^n = \{yz^{n-m+1} : yz \in X^m\}$  if  $m \leq n$ . It follows that for every  $m \geq 1$  we have  $X_m^n \subseteq X_{m+1}^n \subseteq X^n$ .

**Definition 9.** Let  $m \geq 1$  and  $n \geq 0$  be integers. We say that a function  $H: X^n \rightarrow X^n$  has an  $m$ -generated range if  $\text{ran}(H) \subseteq X_m^n$ . We say that a function  $H: X^* \rightarrow X^*$  has an  $m$ -generated range if  $H_n$  has an  $m$ -generated range for every  $n \geq 0$ .

**Fact 10.** *If a function  $H: X^n \rightarrow X^n$  has an  $m$ -generated range, then it has an  $(m+1)$ -generated range. If a function  $H: X^* \rightarrow X^*$  has an  $m$ -generated range, then it is length-preserving.*

Let  $m \geq 1$  and  $n \geq 0$  be integers. The  $m$ -diagonal section of a function  $F: X^n \rightarrow Y$  is the map  $\delta_F^m: X^{\min\{n,m\}} \rightarrow Y$  defined by  $\delta_F^m = F$ , if  $n = 0$ , and  $\delta_F^m(yz) = F(yz^{n-\min\{n,m\}+1})$  for every  $yz \in X^{\min\{n,m\}}$ , otherwise. We clearly have  $\text{ran}(\delta_F^m) \subseteq \text{ran}(\delta_F^{m+1})$ .

**Definition 11.** Let  $m \geq 1$  and  $n \geq 0$  be integers. We say that a function  $F: X^n \rightarrow Y$  is  $m$ -quasi-range-idempotent if  $\text{ran}(F) = \text{ran}(\delta_F^m)$ .

By definition, any  $m$ -quasi-range-idempotent function  $F: X^n \rightarrow Y$  is  $(m+1)$ -quasi-range-idempotent. We also observe that the property of being  $m$ -quasi-range-idempotent is preserved under left composition with unary maps: if  $F: X^n \rightarrow Y$  is  $m$ -quasi-range-idempotent, then so is  $g \circ F$  for any map  $g: Y \rightarrow Y'$ , where  $Y'$  is a nonempty set.

**Proposition 12.** *If  $F: X^* \rightarrow X^*$  is associative and  $F_k$  has an  $m$ -generated range for some  $k, m \geq 1$ , then for any integer  $p \geq 0$  the function  $F_{k+p}$  is  $(m+p)$ -quasi-range-idempotent. In particular,  $F_k$  is  $m$ -quasi-range-idempotent.*

*Proof.* Let  $\mathbf{x} \in X^p$  and  $\mathbf{x}' \in X^k$ . Then, there exists  $\mathbf{yz} \in X^{\min\{k,m\}}$  such that

$$\begin{aligned} F_{k+p}(\mathbf{xx}') &= F_{k+p}(\mathbf{x}F_k(\mathbf{x}')) = F_{k+p}(\mathbf{xyz}^{k-\min\{k,m\}+1}) \\ &= F_{k+p}(\mathbf{xyz}^{(k+p)-\min\{k+p,m+p\}+1}) = \delta_{F_{k+p}}^{m+p}(\mathbf{xyz}), \end{aligned}$$

which shows that  $\text{ran}(F_{k+p}) \subseteq \text{ran}(\delta_{F_{k+p}}^{m+p})$ . The converse inclusion is obvious.  $\square$

**Lemma 13.** *Let  $m, n \geq 1$  be integers. Any map  $F: X^n \rightarrow Y$  satisfying  $F = F \circ H$ , where  $H: X^n \rightarrow X^n$  has an  $m$ -generated range, is  $m$ -quasi-range-idempotent.*

*Proof.* Since  $\text{ran}(H) \subseteq X_m^n$ , we have  $\text{ran}(F) = \text{ran}(F \circ H) \subseteq \text{ran}(\delta_F^m)$ . Since the converse inclusion  $\text{ran}(F) \supseteq \text{ran}(\delta_F^m)$  holds for any map  $F: X^n \rightarrow Y$ , we have that  $F$  is  $m$ -quasi-range-idempotent.  $\square$

**Lemma 14.** *Under the assumptions of Lemma 6, if  $F$  is  $m$ -quasi-range-idempotent for some  $m \geq 1$ , then  $g$  can always be chosen so that  $\text{ran}(g) \subseteq X_m^n$  and therefore  $H$  has an  $m$ -generated range. Conversely, if  $H$  has an  $m$ -generated range for some  $m \geq 1$ , then  $F$  is  $m$ -quasi-range-idempotent.*

*Proof.* If  $F$  is  $m$ -quasi-range-idempotent for some  $m \geq 1$ , then there always exists  $g \in Q(F)$  such that  $\text{ran}(g) \subseteq X_m^n$ ; indeed, if  $y \in \text{ran}(F) = \text{ran}(\delta_F^m)$ , then we can take  $g(y) \in (\delta_F^m)^{-1}\{y\} \subseteq X_m^n$ . Therefore  $H = g \circ F$  has an  $m$ -generated range. Conversely, if  $H$  has an  $m$ -generated range for some  $m \geq 1$ , then  $F$  is  $m$ -quasi-range-idempotent by Lemma 13.  $\square$

**Corollary 15.** *For any  $m \geq 1$ , the equivalence in Lemma 7 holds if we add the condition that every  $F_n$  ( $n \geq 1$ ) is  $m$ -quasi-range-idempotent in assertion (i) and the conditions that  $\text{ran}(g_n) \subseteq X_m^n$  ( $n \geq 1$ ) and  $H$  has an  $m$ -generated range in assertions (ii) and (iii).*

**Theorem 16.** *For any  $m \geq 1$ , the equivalence between (i) and (ii) in Theorem 8 still holds if we add the condition that every  $F_n$  ( $n \geq 1$ ) is  $m$ -quasi-range-idempotent in assertion (i) and the condition that  $H$  has an  $m$ -generated range in assertion (ii). In this case the condition  $\text{ran}(g_n) \subseteq X_m^n$  ( $n \geq 1$ ) must be added in the last part of the statement.*

*Proof.* Follows from the results above.  $\square$

Setting  $m = 1$  in Theorem 16, we immediately derive a factorization of any B-preassociative function whose  $n$ -ary part  $F_n$  is 1-quasi-range-idempotent for every  $n \geq 1$ . An alternative factorization for such functions is given in the following theorem, established in [8]. Recall that a function  $F: X^* \rightarrow X \cup \{\varepsilon\}$  is *barycentrically associative* (or *B-associative* for short) [8] if it satisfies the equation  $F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$  for any  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ . (B-associativity is also known as *decomposability*, see [2, 3]).

**Theorem 17** ([8]). *Assume AC and let  $F: X^* \rightarrow Y$  be a function. The following assertions are equivalent.*

- (i)  *$F$  is B-preassociative and  $F_n$  is 1-quasi-range-idempotent for every  $n \geq 1$ .*
- (ii) *There exists a B-associative function  $H: X^* \rightarrow X \cup \{\varepsilon\}$  such that  $H(\varepsilon) = \varepsilon$  and a sequence  $(f_n)_{n \geq 1}$  of one-to-one functions  $f_n: \text{ran}(H_n) \rightarrow Y$  such that  $F_n = f_n \circ H_n$  for every  $n \geq 1$ .*

*If condition (ii) holds, then for every  $n \geq 1$  we have  $F_n = \delta_{F_n} \circ H_n$ ,  $f_n = \delta_{F_n}|_{\text{ran}(H_n)}$ ,  $f_n^{-1} \in Q(\delta_{F_n})$ , and we may choose  $H_n = g_n \circ F_n$  for any  $g_n \in Q(\delta_{F_n})$ .*

We now show how Theorem 17 can be easily derived from Theorem 16.

For every  $m \geq 1$  and every  $\mathbf{x} \in X^*$ , denote by  $\mathbf{x}_{[m]}$  the  $m$ -prefix of  $\mathbf{x}$ , that is the string in  $\bigcup_{i=0}^m X^i$  defined as follows: if  $|\mathbf{x}| \leq m$ , then  $\mathbf{x}_{[m]} = \mathbf{x}$ ; otherwise, if  $\mathbf{x} = \mathbf{x}'\mathbf{x}''$ , with  $|\mathbf{x}'| = m$ , then  $\mathbf{x}_{[m]} = \mathbf{x}'$ .

If  $H: X^* \rightarrow X^*$  has an  $m$ -generated range, then by definition it can be assimilated with the function  $H_{[m]}: X^* \rightarrow \bigcup_{i=0}^m X^i$  defined by  $H_{[m]}(\mathbf{x}) = H(\mathbf{x})_{[m]}$ . Indeed,  $H$  can be reconstructed from  $H_{[m]}$  by setting

$$H(\mathbf{x}) = \begin{cases} H_{[m]}(\mathbf{x}), & \text{if } |\mathbf{x}| \leq m, \\ H_{[m]}(\mathbf{x})z^{n-m}, & \text{otherwise,} \end{cases}$$

where  $z$  is the last letter of  $H_{[m]}(\mathbf{x})$ .

Thus we can prove Theorem 17 from Theorem 16 as follows.

*Proof of Theorem 17 as a corollary of Theorem 16.* By setting  $m = 1$  in Theorem 16, we see that  $H$  has a 1-generated range. By the observation above,  $H$  can then be assimilated with  $H_{[1]}$  through the identity  $H(\mathbf{x}) = H_{[1]}(\mathbf{x})^{|\mathbf{x}|}$  for every  $\mathbf{x} \in X^*$ . It is then clear that  $H$  is associative if and only if  $H_{[1]}$  is B-associative. The other parts of Theorem 17 follow immediately.  $\square$

**Remark 4.** The question of generalizing Theorem 17 by dropping the 1-quasi-range-idempotent condition on every  $F_n$  was raised in [8]. Clearly, Theorem 8 answers this question.

### 3. SOME CONSEQUENCES OF THE FACTORIZATION RESULT

Since any associative function  $F: X^* \rightarrow X^*$  is preassociative and, in turn, B-preassociative, it can be factorized as indicated in Theorem 8. Therefore, up to one-to-one unary maps, the associative string functions can be completely described in terms of length-preserving associative string functions, and similarly for the preassociative and B-preassociative functions. This is an important observation which shows that in a sense any of these nested classes can be described in terms of the smallest one, namely the subclass of associative and length-preserving string functions (see Figure 1).

**Example 18.** Let  $a \in X$  be fixed. Let the map  $F: X^* \rightarrow X^*$  be defined inductively by  $F(z) = z$  if  $z \neq a$ ,  $F(a) = \varepsilon$ , and  $F(\mathbf{x}z) = F(\mathbf{x})F(z)$  for every  $\mathbf{x}z \in X^*$ . Thus defined,  $F(\mathbf{x})$  is obtained from  $\mathbf{x}$  by removing all the ‘a’ letters (if any). Since  $F$  is

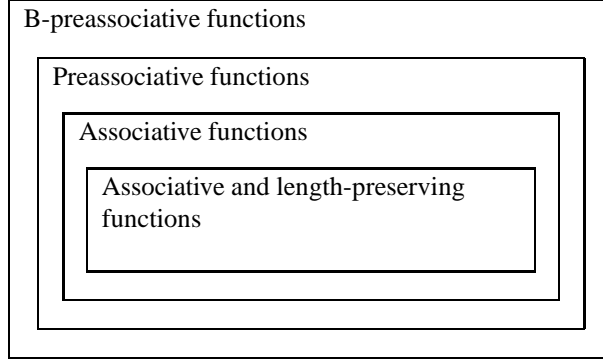


FIGURE 1. Nested subclasses of B-preassociative functions

associative (see [5] for more details), it is B-preassociative and therefore it can be factorized as indicated in Theorem 8. For every  $n \geq 1$ , define the function  $g_n: \bigcup_{i=0}^n (X \setminus \{a\})^i \rightarrow X^n$  by  $g_n(\mathbf{x}) = \mathbf{x}a^{n-|\mathbf{x}|}$ . Since  $F_n \circ g_n \circ F_n = F_n$  for every  $n \geq 1$ , we see that  $g_n \in Q(F_n)$ . By Theorem 8, the function  $H: X^* \rightarrow X^*$ , defined by  $H_0(\varepsilon) = \varepsilon$  and  $H_n = g_n \circ F_n$  for every  $n \geq 1$ , is associative and length-preserving. Moreover, we have  $F_n = f_n \circ H_n$  for every  $n \geq 1$ , where  $f_n = F_n|_{\text{ran}(H_n)}$ . Thus defined,  $H_n(\mathbf{x})$  is obtained from  $\mathbf{x}$  by moving all the ‘a’ letters (if any) to the rightmost positions. For instance,  $H_{11}(\text{mathematics}) = \text{mthemticsaa}$ .

As observed in the previous section, setting  $m = 1$  in Theorem 16, we can derive a factorization of any B-preassociative function whose  $n$ -ary part  $F_n$  is 1-quasi-range-idempotent for every  $n \geq 1$  (Theorem 17). In the following example, we derive a similar factorization explicitly directly from Theorem 8 (without using Theorem 16).

**Example 19.** If we assume that  $F_n$  is 1-quasi-range-idempotent for every  $n \geq 1$  in assertion (i) of Theorem 8, then the factorization given in assertion (ii) can be obtained by defining  $H_n = g_n \circ F_n$ , where  $g_n(x) = h_n(x)^n$  and  $h_n \in Q(\delta_{F_n})$ . Indeed, since  $F_n$  is 1-quasi-range-idempotent, we have

$$(F_n \circ g_n \circ F_n)(\mathbf{x}) = (\delta_{F_n} \circ h_n \circ F_n)(\mathbf{x}) = F_n(\mathbf{x}),$$

which shows that  $g_n \in Q(F_n)$ .

It is clear that the B-associativity property, originally defined for functions  $F: X^* \rightarrow X \cup \{\varepsilon\}$  can be immediately extended to string functions  $F: X^* \rightarrow X^*$ .

**Definition 20.** We say that a string function  $F: X^* \rightarrow X^*$  is *barycentrically associative* (or *B-associative* for short) if it satisfies the equation  $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})^{|\mathbf{z}|})$  for any  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ .

It is easy to see that any B-associative string function  $F: X^* \rightarrow X^*$  is B-preassociative and hence can be factorized as indicated in Theorem 8. Moreover, any B-associative string function satisfying  $\text{ran}(F_n) \subseteq X$  for every  $n \geq 1$  is also such that  $F_n$  is 1-quasi-range-idempotent for every  $n \geq 1$  (see [8]) and therefore it can be factorized as described in Example 19. In this case we have  $\delta_{F_n} \circ F_n = F_n$ , which shows that  $\text{id}_{|\text{ran}(F_n)|} \in Q(\delta_{F_n})$  for every  $n \geq 1$ . Therefore, from Example 19 we immediately derive the following corollary.

**Corollary 21.** *Let  $F: X^* \rightarrow X^*$  be a B-associative function satisfying  $\text{ran}(F_n) \subseteq X$  for every  $n \geq 1$ . Then, for every  $n \geq 1$ , we have  $F_n = f_n \circ H_n$ , where  $H: X^* \rightarrow X^*$  is the length-preserving associative function defined by  $H_n(\mathbf{x}) = F_n(\mathbf{x})^n$  for every  $n \geq 1$  and  $f_n: \text{ran}(H_n) \rightarrow X$  is the one-to-one function defined by  $f_n(x^n) = x$  for every  $n \geq 1$ .*

We end this section by an additional application of Theorem 8.

**Definition 22.** We say that a function  $F: X^* \rightarrow Y$  has a *componentwise defined kernel* if there exists a family  $\{E_n : n \geq 1\}$  of equivalence relations on  $X$  such that for any  $n \geq 1$  and any  $\mathbf{x}, \mathbf{y} \in X^n$ , we have  $F(\mathbf{x}) = F(\mathbf{y})$  if and only if  $(x_i, y_i) \in E_i$  for  $i = 1, \dots, n$ . In this case, we say that the family  $\{E_n : n \geq 1\}$  *defines the kernel of  $F$  componentwise*.

This concept can be interpreted, e.g., in decision making, as follows. A function  $F: X^* \rightarrow Y$  has a componentwise defined kernel if the equivalence between two  $n$ -profiles  $\mathbf{x}, \mathbf{y} \in X^n$  can be defined attributewise.

The following proposition and corollary give characterizations of those B-preassociative functions which have a componentwise defined kernel.

**Proposition 23.** *Assume AC and let  $F: X^* \rightarrow Y$  have a kernel defined componentwise by the family  $\{E_n : n \geq 1\}$  of equivalence relations on  $X$ . Then  $F$  is B-preassociative if and only if  $E_n \subseteq E_{n+1}$  for every  $n \geq 1$ .*

*Proof.* Let  $F: X^* \rightarrow Y$  be defined as indicated in the statement. For the necessity, suppose that  $F$  is B-preassociative and let  $(x, y) \in E_n$  for some  $n \geq 1$ . Then we have  $F(x^n) = F(x^{n-1}y)$  and hence  $F(x^{n+1}) = F(x^n y)$  by B-preassociativity. It follows that  $(x, y) \in E_{n+1}$ . For the sufficiency, for any  $n \geq 1$  and any  $\mathbf{x}, \mathbf{y} \in X^n$  such that  $F(\mathbf{x}) = F(\mathbf{y})$ , we have  $F(\mathbf{xz}) = F(\mathbf{yz})$  for every  $\mathbf{z} \in X^*$  by definition of  $F$ . Since  $E_n \subseteq E_{n+1}$  for every  $n \geq 1$ , we also have  $F(\mathbf{zx}) = F(\mathbf{zy})$  for every  $\mathbf{z} \in X^*$ . Therefore  $F$  is B-preassociative.  $\square$

**Corollary 24.** *Assume AC and let  $F: X^* \rightarrow Y$  be a function. The following assertions are equivalent.*

- (i)  *$F$  is B-preassociative and has a componentwise defined kernel.*
- (ii) *There exists a sequence  $(h_n)_{n \geq 1}$  of unary operations on  $X$  and a sequence  $(f_n)_{n \geq 1}$  of one-to-one maps  $f_n: \{h_1(x_1) \cdots h_n(x_n) : x_1 \cdots x_n \in X^n\} \rightarrow Y$  such that  $h_n \circ h_n = h_n$ ,  $h_{n+1} \circ h_n = h_{n+1}$ , and  $F_n(\mathbf{x}) = f_n(h_1(x_1) \cdots h_n(x_n))$  for every  $n \geq 1$  and every  $\mathbf{x} \in X^n$ .*

*Proof.* (i)  $\Rightarrow$  (ii). By Proposition 23, the kernel of  $F$  is defined by some family of equivalence relations  $\{E_n : n \geq 1\}$  on  $X$  satisfying  $E_n \subseteq E_{n+1}$  for every  $n \geq 1$ . For every  $c \in X/E_n$ , let  $s_n(c) \in c$  be a representative of  $c$  and define the map  $h_n: X \rightarrow X$  by  $h_n(x) = s_n(x/E_n)$ . The map  $g_n: \text{ran}(F_n) \rightarrow X^n$  defined by  $g_n(F(\mathbf{x})) = h_1(x_1) \cdots h_n(x_n)$  is a quasi-inverse of  $F_n$ . Indeed, since  $(x_i, h_i(x_i)) \in E_i$  for every  $\mathbf{x} \in X^n$  and every  $i \in \{1, \dots, n\}$ , we have

$$(F_n \circ g_n \circ F_n)(x_1 \cdots x_n) = F_n(h_1(x_1) \cdots h_n(x_n)) = F_n(x_1 \cdots x_n).$$

By Theorem 8, setting  $H_n = g_n \circ F_n$  for every  $n \geq 1$ , there is a one-to-one function  $f_n: \text{ran}(H_n) \rightarrow Y$  such that  $F_n = f_n \circ H_n$  and such that the map  $H: X^* \rightarrow X^*$  obtained by setting  $H_0(\varepsilon) = \varepsilon$  is associative and length-preserving. The conclusion follows from Example 3.

(ii)  $\Rightarrow$  (i) By Example 3 and Proposition 4 we obtain that  $F$  is B-preassociative. Moreover, the kernel of  $F$  is defined by the family  $\{\ker(h_i) : i \geq 1\}$  of equivalence relations on  $X$ .  $\square$



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